

A topological interpretation of the color charge

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We develop a theory on a topologically non-trivial manifold which leads to different vacuum backgrounds at the field level. The different colors of the same quark flavor live in different backgrounds generated by the action of the torsion subgroup of $H^2(M, 2\pi\mathbb{Z})$ on $H^2(M, 2\pi\mathbb{Z})$ itself. This topological separation leads to a quark confinement mechanism which does not apply to the baryons as they turn out to live on the same vacuum state. For the baryons the usual gauge theory holds but for the quarks a slightly general version should be used. The $U(1)$ sector of the Standard Model is imbedded into the $SU(3)$ sector, the coefficients of the $SU(3)$ gauge transformations being sections of suitable complex line bundles. The theory makes some topological assumptions on the spacetime manifold which are compared with the available data on the topology of the Universe.

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INTRODUCTION

The Standard Model is based on the gauge group $SU(3) \otimes SU(2) \otimes U(1)$. In this work we are going to exploit the relation between the $U(1)$ and the $SU(3)$ sectors in spacetime manifolds M with non-trivial topology. The idea, clarified below, is that each particle can be identified, at the electromagnetic level, with a cohomology class on $H^2(M, 2\pi\mathbb{Z})$. If the topology is not trivial and $H^2(M, 2\pi\mathbb{Z})$ admits a torsion subgroup of order three then there is a natural action of this subgroup on $H^2(M, 2\pi\mathbb{Z})$ which produces orbits with three elements. The three cohomology classes identify three ‘particles’ that behave in the same way but that do not coincide. We identify these three particles with the three colors of the same quark flavor. Thus for every cohomology class on $H^2(M, 2\pi\mathbb{Z})$ we have three equivalent copies which are the different colors of the ‘same’ particle. As a consequence, the cyclic torsion subgroup becomes a subgroup of the $SU(3)$ group acting on colors. In this way we obtain a natural non-trivial embedding of the $U(1)$ sector into an $SU(3)$ covariant structure, an appealing unification that has been regarded as fundamental in order to

reach a deeper understanding of the electric charge quantization and quark confinement [7, 14].

Remarkably, the possibility considered here has cosmological consequences as it is based on some hypothesis on the topological structure of spacetime. The connection with cosmology is clear under the assumption that the non-trivial topology manifests itself at the largest length scales. Under this assumption our particle model could have been falsified by cosmological observations, and provides an example of interaction between very small scale physics and very large scale physics. These aspects are considered in the last part of the work where it is shown that the latest cosmological data are compatible with our model.

PARTICLES AND COHOMOLOGY CLASSES

For simplicity in what follows we shall ignore the $SU(2)$ sector of the Standard Model. Alternatively the reader can think of the following fields as the right-handed components of the particle considered, since these components in the Standard Model transform with the $SU(2)$ trivial representation. As a consequence of our simplification the $U(1)$ sector can be identified with the electromagnetic sector.

In order to fix the ideas let us focus on the quark flavor \bar{d} of charge $q = e/3$, and on its three colored fields $\bar{d}(r)$, $\bar{d}(g)$ and $\bar{d}(b)$. According to the usual treatment of the $U(1)$ gauge theory [6] the matter fields are sections of a suitable 1-dimensional vector bundle associated with a universal $U(1)$ bundle Q . On the bundle Q a (electromagnetic) connection 1-form ω can be defined and all the fields can be considered as suitable sections of

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vector bundles associated to it through a representation $\rho_n : U(1) \rightarrow \mathbb{C}$ of the form $u \rightarrow u^n$ where nq , $n \in \mathbb{N}$, is the (electric) charge of the particle. Sometimes it can be convenient to regard a particle as a section of a vector bundle associated to the bundle Q^n under the trivial representation ρ_1 . We shall use this point of view as it allows to associate each kind of particle with a certain principal $U(1)$ bundle. For instance, the three colored components of \bar{d} live on the same vector bundle. We denote this fact by writing down the $U(1)$ bundles associated to them

$$\bar{d} : \quad Q, \quad Q, \quad Q. \quad (1)$$

For the antiparticle d and the flavor u we have

$$d : \quad Q^{-1}, \quad Q^{-1}, \quad Q^{-1}, \quad (2)$$

$$u : \quad Q^2, \quad Q^2, \quad Q^2. \quad (3)$$

The manifold M admits a good covering $\{U_i\}$. On each open set U_i there is a local potential 1-form B^i such that on the intersections $U_{ij} = U_i \cap U_j$, $dB^i = dB^j$, that is the curvature 2-form of the $U(1)$ sector is globally well defined. These 1-forms are the pullbacks of the connection 1-form ω under the trivializing sections $\sigma_i : U_i \rightarrow Q$. Let $g_{ij} = \sigma_i \circ \sigma_j^{-1} = e^{i\beta_{ij}}$ be the transition functions. A particle ψ is represented on each set U_i by a field ψ^i that in the intersection U_{ij} transforms as [16]

$$A^i = A^j + d\beta^{ij}, \quad (4)$$

$$\psi^i = e^{in\beta^{ij}} \psi^j. \quad (5)$$

In particular the colored components associated with the same quark flavor, say u , transform in the same way

$$u^i(r) = e^{i2\beta^{ij}} u^j(r), \quad (6)$$

$$u^i(g) = e^{i2\beta^{ij}} u^j(g), \quad (7)$$

$$u^i(b) = e^{i2\beta^{ij}} u^j(b). \quad (8)$$

However, this formalism although successful does not accommodate in a simple way the observational fact that there are actually two fundamental electric charges, say the fundamental unit q , and the fundamental unit for baryons and leptons $e = 3q$.

Consider the short exact sequence

$$0 \rightarrow 2\pi\mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{\exp(ix)} U(1) \rightarrow 1, \quad (9)$$

where we identify $U(1)$ and $\mathbb{R}/2\pi\mathbb{Z}$. It gives rise to the long exact sequence

$$\begin{aligned} \dots \rightarrow H^1(M, 2\pi\mathbb{Z}) &\rightarrow H^1(M, \mathbb{R}) \xrightarrow{\sigma} \\ H^1(M, U(1)) &\xrightarrow{\eta} H^2(M, 2\pi\mathbb{Z}) \xrightarrow{\gamma} H^2(M, \mathbb{R}) \rightarrow \dots, \end{aligned}$$

where $\text{Im}(\eta) = \text{Ker}(\gamma)$ is the torsion subgroup of $H^2(M, 2\pi\mathbb{Z})$. We recall that classes in $H^1(M, U(1))$ can be identified with the isomorphism classes of flat bundles and the torsion classes, but for the zero class, represent those flat bundles which are not trivial.

THE TORSION SUBGROUP

In [10] it has been recognized that the usual gauge principle can be generalized in a way compatible with the Standard Model. In this generalization the transition functions of two different fields may not necessarily be the same as they could differ locally by a constant (weak gauge principle). Thus Eqs. (6)-(8) can be suitably generalized.

In a simply connected spacetime this generalization does not lead to a different physics. Nevertheless, if $H^2(M, 2\pi\mathbb{Z})$ contains a non-vanishing torsion subgroup then a new interesting physics arises. Indeed, let us assume that the torsion is a cyclic subgroup $\langle K \rangle$ of order three (i.e. isomorphic to $\mathbb{Z}/3$) generated under tensor product by a non-trivial flat bundle K . This subgroup acts on $H^2(M, 2\pi\mathbb{Z})$ generating orbits containing three elements.

In our formalism the colored components live on vector bundles associated with different $U(1)$ bundles, the different $U(1)$ bundles being on the orbit generated by the action of $\langle K \rangle$ on $H^2(M, 2\pi\mathbb{Z})$. Thus for instance the colors of the generation (u, d) live in

$$u : \quad Q^2, \quad Q^2 \otimes K, \quad Q^2 \otimes K^2, \quad (10)$$

$$d : \quad Q^{-1}, \quad Q^{-1} \otimes K, \quad Q^{-1} \otimes K^2, \quad (11)$$

for a suitable choice of representant Q on the orbit of the $U(1)$ principal bundles associated to \bar{d} . The fact that there is a freedom in choosing the initial principal bundle Q , manifests itself in a cyclic symmetry generated by the replacements $Q \rightarrow Q \otimes K$. The next flavor generations (c, s) and (t, b) live on the same principal bundles. For instance $c(r)$ is a section on the same vector bundle that corresponds to $u(r)$, i.e. on a vector bundle associated to Q^2 .

To fix the ideas consider the (gauge) transformation of the colored fields for the flavor u from U_j to U_i

$$A^i = A^j + d\beta^{ij}, \quad (12)$$

$$u^i(r) = e^{i2\beta^{ij}} u^j(r), \quad (13)$$

$$u^i(g) = e^{i2\beta^{ij} + ik^{ij}} u^j(g), \quad (14)$$

$$u^i(b) = e^{i2\beta^{ij} + i2k^{ij}} u^j(b). \quad (15)$$

here $e^{ik^{ij}}$ are the constant transition functions of the flat bundle K . Since K^3 is a trivial flat bundle $k^{ij} \in \frac{2\pi}{3}\mathbb{Z}$, up to constant terms $h^i - h^j$ that can be absorbed with a redefinition of the fields on each set.

THE $SU(3)$ SYMMETRY

Let us come to the $SU(3)$ symmetry. The standard treatment regards it as a Lagrangian symmetry under

$SU(3)$ transformations of the (vertical) flavor vectors, for instance for u we have

$$\begin{pmatrix} u'(r) \\ u'(g) \\ u'(b) \end{pmatrix} = \begin{pmatrix} U_{rr} & U_{rg} & U_{rb} \\ U_{gr} & U_{gg} & U_{gb} \\ U_{br} & U_{bg} & U_{bb} \end{pmatrix} \begin{pmatrix} u(r) \\ u(g) \\ u(b) \end{pmatrix}.$$

In our case U can not be a matrix whose coefficients are complex numbers, as $u'(r)$ must live in the same vector bundle as $u(r)$ (and analogously for $u(g)$ and $u(b)$). As a consequence, the components of U must be taken to be sections of vector bundles associated to K^i with the following exponents

$$\begin{pmatrix} K^0 & K^2 & K \\ K & K^0 & K^2 \\ K^2 & K & K^0 \end{pmatrix}.$$

The matrix U becomes a $SU(3)$ matrix only if a local section of K is chosen. At the intersection U_{ij} the trivializing section changes with transition function $e^{ik_{ij}}$, and the complex representant changes as

$$\begin{pmatrix} U_{rr} & U_{rg} & U_{rb} \\ U_{gr} & U_{gg} & U_{gb} \\ U_{br} & U_{bg} & U_{bb} \end{pmatrix} \rightarrow \begin{pmatrix} U_{rr} & U_{rg}e^{i2k^{ij}} & U_{rb}e^{ik^{ij}} \\ U_{gr}e^{ik^{ij}} & U_{gg} & U_{gb}e^{i2k^{ij}} \\ U_{br}e^{i2k^{ij}} & U_{bg}e^{ik^{ij}} & U_{bb} \end{pmatrix}.$$

It is easily seen that a change of section sends $SU(3)$ matrices to $SU(3)$ matrices. In this model the $U(1)$ vector bundles K^i are the building blocks from which the $SU(3)$ invariance is constructed. We mention that, following a different route, the authors of [7, 14] also advocated the need for a unification of the $U(1)$ and $SU(3)$ sectors in a suitable gauge structure. Their strategy differs from ours since, instead of enlarging the gauge group, we change the very definition of $SU(3)$ transformation.

Taking into account that every baryon is a color singlet, we have that the principal bundle associated to it has the form $(Q^3)^k$ for a suitable integer k . In particular, it has charge ke . For instance the proton $p = duu'$ has a field

$$p = d(r)u(g)u'(b) + d(b)u(r)u'(g) + d(g)u(b)u'(r) - d(r)u(b)u'(g) - d(g)u(r)u'(b) - d(b)u(g)u'(r).$$

The terms in the sum are sections of the same vector bundle (associated to Q^3) and for this reason they can be summed up. Indeed, the K factors of the $U(1)$ bundles associated with the different colors cancel in the products. The cancellation is a general consequence of the fact that the cyclic subgroup $\langle K \rangle$ is included into $SU(3)$.

The $U(1)$ principal bundles associated with the baryonic fields have a common root Q^3 which is natural to identify with the $U(1)$ principal bundle root of the principal bundles associated to the leptonic fields. Therefore, in this model, if quarks are not taken into account, the usual gauge theory, with a universal $U(1)$ bundle Q^3 applies.

INITIAL CONDITIONS AND DYNAMICS

Let the spacetime be a direct product between time and space $M = T \times S$, $T = \mathbb{R}$. Since \mathbb{R} is contractible the cohomology groups depend only on the space manifold S , for instance $H^2(M, 2\pi\mathbb{Z}) \simeq H^2(S, 2\pi\mathbb{Z})$. Denote with π_S the projection $\pi_S : M \rightarrow S$ and let $\{V_i\}$ be a good covering of S . It is convenient to take the good covering for M defined by $U_i = \pi^{-1}(V_i)$. The previous isomorphism can be regarded as a consequence of the following

Lemma .1. *Through suitable changes of sections on each U_i one can make the functions β^{ij} of the gauge transformation (12) between each pair U_i, U_j , to be independent of time, $\partial_t \beta^{ij} = 0$.*

Proof. The functions β^{ij} on the triple intersections give through the constant combination $c_{ijk} = \beta^{ij} + \beta^{jk} + \beta^{ki}$ a class $[c_{ijk}]$ on $H^2(M, 2\pi\mathbb{Z})$ (for the details see [10]). Differentiating with respect to time $\partial_t \beta^{ij} + \partial_t \beta^{jk} + \partial_t \beta^{ki} = 0$, which means that $\partial_t \beta_{ij}$ are the transition functions of a principal bundle of fiber \mathbb{R} and group $(\mathbb{R}, +)$. Since the fiber is contractible the bundle is trivial and there are functions $\partial_t \alpha^i$ such that $\partial_t \beta^{ij} + \partial_t \alpha^i - \partial_t \alpha^j = 0$ or $\beta'^{ij} = \beta^{ij} + \alpha^i - \alpha^j$ is independent of time (α^i can be chosen such that $\alpha^i = 0$ at time t so that at the same time $\beta'^{ij} = \beta^{ij}$). Thus a change of section on each U_i with transition function $e^{i\alpha^i}$ makes the functions β^{ij} appearing in (12) independent of time. \square

Note that the information of the class to which the matter field, say $\{\psi^i\}$, belongs in $H^2(M, 2\pi\mathbb{Z})$ must be provided with the initial conditions $\{\psi^i\}_{t=0}$, otherwise the dynamics is in most cases under determined. To see this imagine for instance an initial condition such that the component ψ^i is the only one that differs from zero, and the region where it does not vanish is an open set A such that $A \cap U_j = \emptyset$, for $j \neq i$. Then the equations of dynamics (suppose they are linear for simplicity) on U_i will spread out the region on which ψ^i differs from zero, say $A(t)$, untill $A(t)$ starts having non empty intersections with U_j , $j \neq i$. At this stage in order to continue the evolution of the field, one has to use the dynamics equations on U_j too, and in order to do this, ψ^i must be transformed into ψ^j , a step which requires the knowledge of the transition functions $e^{iq_\psi \beta^{ji}}$ in a given gauge. In particular we may take the gauge in which β^{ij} are constants.

We conclude that if the dynamics satisfies the gauge principle, i.e. the Lagrangian is invariant under weak gauge transformations, then the matter field belongs to a topological sector that must be specified in the initial conditions and which remains unaltered during the evolution of the field. Thus the dynamics can not transform a particle corresponding to a given class of $H^2(S, 2\pi\mathbb{Z})$ into a different class. In our application to the color

of quarks this means that the dynamics preserves the color of a given quark flavor. The class of a particle on $H^2(S, 2\pi\mathbb{Z})$ is a true quantum number and in the interaction of different particles its balance is obtained through the usual tensor product rule for principal bundles.

In general, given a principal $U(1)$ bundle $P \in H^2(M, 2\pi\mathbb{Z})$, the principal bundles on the orbit generated by the action of $\langle K \rangle$,

$$P, \quad P \otimes K, \quad P \otimes K^2, \quad (16)$$

represent different topological (vacuum) sectors of the theory. In particular, we may consider the hypothesis that the transition between them requires a huge amount of energy. As a consequence of this assumption all the baryons, sharing a common square root Q^3 would live in the same topological vacuum state, while the isolated quarks would not. The interaction between baryons would follow the usual rules of the Standard Model but the free quarks would not be seen as the energy required to move the state from one topological sector to the other would be too high. Examples of this topological mechanism are familiar in physics, they run from the simple kink-antikink example to the physics of instantons [13].

THE TOPOLOGY OF THE UNIVERSE

The present theory works only in a spacetime manifold such that $H^2(M, 2\pi\mathbb{Z})$ contains a cyclic torsion subgroup of order three. This fact could provide a constraint for current investigations [1, 5] on the space topology of the Universe under the assumption $M = S \times \mathbb{R}$. Unfortunately, the recently suggested [2, 9] Poincaré dodecahedral topology, $S = \mathcal{P} = S^3/\Gamma_{\mathcal{P}}$ does not share this property since $H^2(\mathcal{P}, 2\pi\mathbb{Z}) = 0$.

The lens space $S = L(3, p)$, for $p = 1, 2$ ($L(3, 2)$ is the ‘mirror image’ of $L(3, 1)$) is obtained from $SU(2) \sim S^3$ by making the quotient with respect to the action of

$$\begin{pmatrix} \omega & 0 \\ & \omega^p \end{pmatrix},$$

with $\omega \in \mathbb{C}$, $\omega^3 = 1$. It provides an example of a homogeneous space compatible with the required cohomology. Indeed, using results from [12], we find

$$H^j(L(3, p), 2\pi\mathbb{Z}) \simeq \begin{cases} 2\pi\mathbb{Z}, & \text{if } j = 0 \text{ or } j = 3, \\ 2\pi\mathbb{Z}/3, & \text{if } j = 2, \\ 0, & \text{otherwise.} \end{cases}$$

If the topological scale of the universe, represented by the injectivity radius r_{inj} , is smaller than the radius of the surface of last scattering χ_{LSS} , then the non-trivial topology of the universe would be detectable using the CMB-circles in the sky method [3]. Both length scales have been studied in deep for many candidate topologies

[11]. In the case of the lens space $L(3, 1)$ it has been found that the injectivity radius is larger than the radius of the last scattering surface as derived from latest cosmological data. Indeed, $L(3, 1)$ is a single action manifold [8] with a cyclic holonomy group of order $n = 3$ which implies [15], $r_{inj} = \pi/n$. In [15] it has been shown that in this case $\chi_{LSS} > r_{inj}$, only if $\Omega_{tot} \gtrsim 1 + 1/n^2$. Present estimates of Ω_{tot} give $\Omega_{tot} = 1.02 \pm 0.02$, which rules out the possibility of detecting the cosmic topology in the $L(3, 1)$ case. The last search of pairs of matching circles in the sky has given a null result [4] and is therefore compatible with a lens shaped Universe $L(3, 1)$.

CONCLUSIONS

In a topologically non-trivial manifold it is possible to weaken the gauge principle allowing for $U(1)$ gauge transformations that only coincide locally for each field up to a constant field-dependent phase. In this way the flat bundles over the manifold become relevant and two cohomology classes on $H^2(M, 2\pi\mathbb{Z})$, i.e. two particles, may differ by the tensor product with a (flat) torsion class. Here we have investigated the example of a torsion subgroup of order three identifying the orbits of the torsion subgroup with the colors of the same quark flavor. This approach has led to an imbedding of the $U(1)$ sector into the $SU(3)$ sector in which the coefficients of the $SU(3)$ transformations are not complex numbers but rather sections of complex line bundles associated to suitable $U(1)$ flat bundles.

The theory, as much as baryons and leptons are concerned, coincides with the usual Standard Model gauge theory. Indeed, we have shown that the baryons live on the same topological background Q^3 while the free quarks live in different topological backgrounds. We have argued that a high transition energy between those backgrounds would give a natural mechanism for the quark confinement. The theory naturally embodies two charge quantization units which can be identified with (i) (minus) the electron charge e , which turns out to be the fundamental charge for those particles that live in the same background (baryons and leptons), and (ii) the electric quantization unit $e/3$.

The theory makes hypothesis on the torsion subgroup of $H^2(M, 2\pi\mathbb{Z})$ and hence on the topology of the Universe. As a working assumption we assumed that the non-trivial topology manifests itself at the cosmological scale. Thus we tested the simplest homogeneous space which satisfies our topological constraint, i.e. the lens space $L(3, 1)$. Although according to some authors it is not favored [9], this space has not yet been ruled out by the latest observations [4]. In any case the lens space $L(3, 1)$ represents only the simplest possibility compatible with this theory. More work is needed to extend our study to more general manifolds which satisfy the

topological constraint.

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- [1] Aurich, R., Lustig, S., and Steiner, F.: CMB anisotropies of spherical spaces. *Class. Quantum Grav.* **22**, 3443–3459 (2005)
 - [2] Aurich, R., Lustig, S., and Steiner, F.: CMB anisotropy of the Poincaré dodecahedron. *Class. Quantum Grav.* **22**, 2061–2083 (2005)
 - [3] Cornish, N. J., Spergel, D. N., and Starkman, G. D.: Circles in the sky: finding topology with the microwave background radiation. *Class. Quantum Grav.* **15**, 2657–2670 (1998)
 - [4] Cornish, N. J., Spergel, D. N., Starkman, G. D., and Komatsu, E.: Constraining the topology of the universe. *Phys. Rev. Lett.* **92**, 201302 (2004)
 - [5] Gomero, M. J. R. G. I.: Cosmic topology: A brief overview. *Braz. J. Phys.* **34**, 1358–1366 (2004)
 - [6] Kobayashi, S. and Nomizu, K.: *Foundations of Differential Geometry*, vol. I of *Interscience tracts in pure and applied mathematics*. New York: Interscience Publishers (1963)
 - [7] Lazarides, G., Shafi, Q., and Trower, W. P.: Consequences of a monopole with Dirac magnetic charge. *Phys. Rev. Lett.* **49**, 1756–1758 (1982)
 - [8] Lehoucq, E. G. R. and Luminet, J.-P.: Topological lensing in spherical spaces. *Class. Quantum Grav.* **18**, 5155–5186 (2001)
 - [9] Luminet, J.-P., Weeks, J., Riazuelo, A., Lehoucq, R., and Uzan, J.-P.: Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background. *Nature* **425**, 593–595 (2003)
 - [10] Minguzzi, E., Prieto, C. T., and Almorox, A. L.: Weak gauge principle and electric charge quantization. *J. Phys. A* **39**, 9591–9610 (2006)
 - [11] Mota, B., Reboucas, M., and Tavakol, R.: Constraints on the detectability of cosmic topology from observational uncertainties. *Class. Quantum Grav.* **20**, 4837–4850 (2003)
 - [12] Munkres, J. R.: *Elements of algebraic topology*. Menlo Park, CA: Addison-Wesley Publishing Company (1984)
 - [13] Rajaraman, R.: *Solitons and instantons*. Amsterdam: North-Holland Publishing Co. (1982)
 - [14] 't Hooft, G.: Magnetic charge quantization and fractionally charged quarks. *Nucl. Phys. B* **105**, 538–547 (1976)
 - [15] Weeks, J., Lehoucq, R., and Uzan, J.-P.: Detecting topology in a nearly flat spherical universe. *Class. Quantum Grav.* **20**, 1529–1542 (2003)
 - [16] We have removed the fundamental charge q from this transformation by absorbing it into the definition of gauge potential, $qA \rightarrow A$.